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Solvable three-state model of a driven double-well potential and coherent destruction of tunneling

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A simple model for a particle in a double well is derived from discretizing its configuration space. The model contains as many free parameters as the original system and it respects all the existing symmetries. In the presence of an external periodic force both the continuous system and the discrete model are shown to possess a generalized time-reversal symmetry in addition to the known generalized parity. The impact of the driving force on the spectrum of the Floquet operator is studied. In particular, the occurrence of degenerate quasienergies causing coherent destruction of tunneling is discussed—to a large extent analytically—for arbitrary driving frequencies and barrier heights. [S1050-2947(98)00301-1]

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I. INTRODUCTION

Tunneling of a particle in a symmetric double-well potential on the real line is well understood, at least qualitatively. For a large barrier separating the minima, semiclassical calculations [1] provide reliable estimates of the system's eigenstates and energy eigenvalues near the bottom of the spectrum. In another approach, path integrals are used to obtain approximate solutions [2,3]. In this framework instantons are crucial; they are the solutions of the classical equations of motion of a particle in the *inverted* potential. For low barriers, one can resort to the supersymmetric partner of the original potential in order to determine the low lying energy eigenvalues [4–6]. The partner potential possesses the *same* spectrum as the original one except for the ground state but, fortunately, it is a *single*-well potential. The approximate evaluation of the spectrum is then straightforward using again semiclassical approximations or, for example, a variational principle.

If an external driving force is added to the system it becomes more difficult to gain insight into its quantum mechanical properties. Even a coupling linear in the particle's coordinate leads to qualitative changes which are not easily discussed in the familiar language of tunneling phenomena [7–10]. To a large extent, this is related to the fact that generically classical systems with one degree of freedom become nonintegrable as a driving force is turned on.

In this paper, an elementary model to describe a particle in a driven double-well system is introduced. The basic idea is to *discretize* the continuous configuration space of the original system while preserving its essential features. In this way, a three-level system is obtained which has both the same number of free parameters and the same symmetries as its continuous ancestor, contrary to existing *two*-level approximations of a double well [11–14]. Many calculations can be performed analytically in this model. The discussion of the undriven system shows that the drastic approximation provides a reasonable qualitative description of the particle's behavior in a double well. On this basis, the dynamics of the driven system is investigated. *Mutatis mutandis*, the results obtained here apply to the continuous system the study of which often requires extensive numerical work.

The paper is organized as follows. First, we briefly review the behavior of a particle in a double well, followed by the derivation of the model to be investigated in this work. The next section discusses the symmetries of the system, formulated in terms of the Floquet description. Then, the properties of the undriven model are compared with those of the continuous system. The driven system is studied in detail, the focus being on degeneracies of quasienergies. Finally, the effective Hamiltonian for time translation over one period is determined approximatively in the high-frequency limit. The summary collects the results and draws conclusions.

II. MODEL OF A DRIVEN DOUBLE WELL

A. Continuous system

The quantal dynamics of a particle in a symmetric double-well potential on the real line,

$$V_{\mathrm{DW}}(x) = -\frac{A}{2}x^2 + \frac{B}{4}x^4, \quad A,B>0 \tag{1}$$

is governed by Schrödinger's equation

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle, \tag{2}$$

where the Hamiltonian reads

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + V_{\mathrm{DW}}(\hat{x}) + g(t)\hat{x}, \quad g(t+T) = g(t). \tag{3}$$

Here a periodic driving force $g(t)$ acting on the particle has been added. The shape of the potential

$$V(x,t) = V_{\mathrm{DW}}(x) + g(t)x \tag{4}$$

at time t depends on the parameters A, B, and $g(t)$. Equivalently, one can describe it in terms of the mean energy V_G of

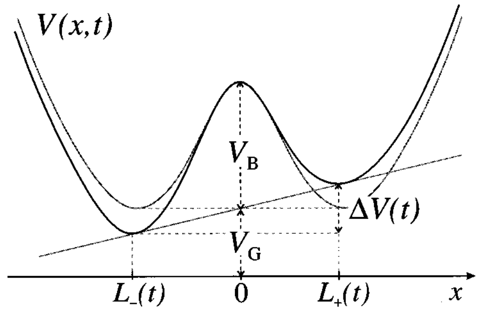

FIG. 1. The double-well potential $V_{DW}(t)$ at time t is characterized by the locations $L_\pm$ of the local minima, the mean energy V_G, the asymmetry ΔV, and the barrier crossing energy V_B.

the potential minima, located at $L_\pm(t)$, the time-dependent asymmetry $\Delta V(t)$, and the barrier height V_B (cf. Fig. 1).

The impact of the driving force on the tunneling behavior of the particle has been studied in [7] for sinusoidal time dependence

$$g(t) = S \sin \omega t, \quad \omega = 2\pi/T. \tag{5}$$

For small and large values of the driving frequency ω, the force has been found to enhance the tunneling rate while the system evolves in a complex manner for intermediate frequencies. In addition, *coherent destruction* of tunneling has been observed [15] for specific parameter values: a wave packet localized initially in one of the wells periodically recovers its shape, even for very long times. This phenomenon can be attributed to a crossing of quasienergies of the Floquet operator. A related phenomenon has been observed in time-independent systems: tunneling is suppressed if energy eigenvalues are forced to fall onto each other as is possible for systems with nontrivial topology in the presence of gauge fields [16,17].

To a large extent, results for a particle in the driven double well are based on numerically obtained solutions of Schrödinger's equation. It seems desirable to have available a model, as simple as possible, which reproduces the features mentioned above at least qualitatively. In the following, an apparently crude approximation of the system described by Eqs. (3) and (5) is introduced. The model is required to respect the qualitative structure of the potential landscape, the number of free parameters in the original system, as well as its symmetries. The resulting model allows one to approach many questions analytically for *all* parameter values. In spite of the drastic simplification involved in its derivation, it is found to provide a reasonable description of tunneling in a driven double-well potential.

B. Discrete system

Let us revert Feynman's "derivation" of Schrödinger's equation [18]: the configuration space $\mathbb{R}$ of the system described by the Hamiltonian $\hat{H}(t)$ is replaced by an equidistant set of points, $x_k = k\Lambda$, with integers k, and the length Λ is yet to be determined. In the position representation the wave function $\psi(x) = \langle x|\psi\rangle$ now takes on values at x_k only,

$$\psi(x) \rightarrow \psi_k = \psi(k\Lambda), \tag{6}$$

and the kinetic energy is proportional to the discretized Laplacian in one dimension [19]:

$$\frac{\partial^2}{\partial x^2}\psi(x) \rightarrow D^2\psi_k = \frac{1}{\Lambda^2}(\psi_{k+1} - 2\psi_k + \psi_{k-1}). \tag{7}$$

As long as the index k runs over all integers, not much has been gained. A further simplification is motivated by looking at the qualitative shape of the potential landscape. A particularly simple, qualitatively correct description of $V_{DW}(x)$ refers only to the presence of the two minima, separated by a barrier at the origin, and to the steep increase for $x \rightarrow \pm\infty$. Therefore the choice $\Lambda = L = |L_\pm(0)|$ allows one to correctly represent the overall structure of $V_{DW}(x)$ if for $x_0 = 0$ and $x_\pm = \pm L$ one defines

$$V(x_0, t) = V_G + V_B, \quad V(x_\pm, t) = V_G \pm \Delta V(t)/2. \tag{8}$$

This approximation does not take into account that the locations of the minima at $x = L_\pm(t)$ are slightly shifted due to the position-dependent driving force.

For simplicity, the boundary condition $\psi(x) \rightarrow 0$ for $x \rightarrow \infty$ is modeled by the requirement that the wave function vanish for $k = \pm 2$, or $\psi_{\pm 2} = 0$, corresponding to an infinitely strong potential $V_{DW}(x_{\pm 2})$ at these points. Thus the wave function is different from zero at three points only, and the discretized version of the Hamiltonian $\hat{H}(t)$ becomes a (3×3) matrix:

$$\mathsf{H}(t) = \mathsf{H}_0 + \mathsf{H}_1(t), \tag{9}$$

where the time-independent part H_0 reads

$$\mathsf{H}_0 = (2\eta + V_G)\mathsf{1} + \begin{pmatrix} 0 & -\eta & 0 \\ -\eta & V_B & -\eta \\ 0 & -\eta & 0 \end{pmatrix}, \tag{10}$$

with $\mathsf{1}$ being the (3×3) unit matrix, and $\eta = \hbar^2/2mL^2$. Here and below, operators acting on the Hilbert space $\mathbb{C}^3$ of the three-state model are denoted by sans-serif symbols. The driving term is

$$\mathsf{H}_1(t) = \frac{\Delta V(t)}{2}\operatorname{diag}(-1,0,1), \tag{11}$$

with a periodically varying asymmetry [cf. Eq. (5)]

$$\Delta V(t) = L g(t). \tag{12}$$

Further, the sinusoidal time dependence of the driving term is replaced by a function taking two values only, being constant during both the first and the second half of the interval of periodicity $T = 2\pi/\omega$:

$$\Delta V(t) = \begin{cases} +\Delta V, & 0 \leq t \mod T < T/2 \\ -\Delta V, & T/2 \leq t \mod T < T. \end{cases} \tag{13}$$

This simplification retains the relevant features of the continuously varying time dependence, as is known, for example, from investigations of parametric resonance [20,21].

Let us check the number of parameters in the discrete model of the periodically driven double well. The Hamiltonian $\mathsf{H}(t)$ in Eq. (9) depends on four parameters, namely the barrier height V_B, the kinetic parameter $\eta=\hbar^2/2mL^2$, the period T of the driving, and its amplitude ΔV, matching thus the number of parameters of the continuous system. If the double-well potential is described by a system with two states (instead of three), the number of parameters is reduced by one, since the model now depends on a combination $\Omega(\eta,V_B)$ of the barrier height V_B and the parameter η only. In this sense, the three-state model introduced here is more realistic than two-level models known to reproduce qualitatively various aspects of the continuous system [14].

Nevertheless, it is useful to eliminate the kinetic parameter η by rescaling

$$T\rightarrow\eta T,\quad \Delta V\rightarrow\Delta V/\eta,\quad V_B\rightarrow V_B/\eta, \tag{14}$$

leaving us with a system depending effectively only on the three (rescaled) parameters V_B, T, and ∇V.

The Hamiltonian $\mathsf{H}(t)$ obtained as an approximation to a particle system also has an interpretation as a Hamiltonian for a spin of length $s=1$ in a crystal field under the influence of a time-dependent external magnetic field along the z axis:

$$\mathsf{H}(t)=-V_B\mathsf{S}_z^2-g\mu_B\mathbf{B}(t)\cdot\mathbf{S}, \tag{15}$$

where a term proportional to the unit matrix has been dropped, and

$$g\mu_B\mathbf{B}(t)=\frac{\eta}{\sqrt{2}}e_x+\Delta V(t)e_z. \tag{16}$$

The vector $\mathbf{S}$ has three components, each of which is a $(2s+1=)$ three-dimensional matrix such that

$$[\mathsf{S}_j,\mathsf{S}_k]=i\varepsilon_{jkl}\mathsf{S}_l. \tag{17}$$

The extrema of the double-well potential correspond to the stationary configurations of the classical spin. The off-diagonal elements of $\mathsf{H}(t)$ couple the stationary states.

III. SYMMETRIES

Before investigating the dynamics of either $\hat{H}(t)$ or $\mathsf{H}(t)$ in detail, a careful search for symmetry transformations is useful. For the class of systems studied here, three independent discrete transformations can be identified which leave the Hamiltonian invariant. Two of them are immediately recognized, namely, the time periodicity and a generalized parity transformation. The third one, a generalization of time-reversal invariance, has not yet been pointed out. The existence of these symmetries implies that the solutions of the driven system have specific features, and each of the invariances simplifies its study.

A. Time periodicity

Due to Eq. (5) or Eq. (13) the Hamilton operator $\hat{H}(t)$ is invariant under the discrete transformation

$$\Xi:t\rightarrow t+T. \tag{18}$$

The long-time properties of a system with period T are conveniently extracted from a description in terms of its *Floquet* operator [22]. Mathematically, the description of electrons in a spatially periodic potential and the Floquet formalism are closely related. Physically, it can be thought of as a stroboscopic observation of the system at times $t=0,T,2T,\ldots$, say. The details of the time evolution for intermediate times are not determined. This approach comes down to studying the properties of the propagator over one time interval

$$\begin{aligned}\hat{\mathcal{F}}&=\hat{U}(T,0)\\&=\lim_{N\to\infty}e^{-(i/\hbar)\hat{H}(t_N)\Delta t}\ldots e^{-(i/\hbar)\hat{H}(t_2)\Delta t}e^{-(i/\hbar)\hat{H}(t_1)\Delta t},\end{aligned} \tag{19}$$

where N is the number of time intervals of length $\Delta t=T/N$, and $t_n=(n-1/2)\Delta t$. Formally, the propagator can be written as

$$\hat{\mathcal{F}}=\tau\exp\left(-\frac{i}{\hbar}\int_0^T dt\ \hat{H}(t)\right), \tag{20}$$

where τ denotes time ordering. The Floquet operator $\hat{\mathcal{F}}$ maps a state $|\psi(0)\rangle$ over one time interval:

$$\hat{\mathcal{F}}|\psi(0)\rangle=|\psi(T)\rangle, \tag{21}$$

corresponding to an integration of Schrödinger's equation from $t=0$ to $t=T$. The N-fold application of $\hat{\mathcal{F}}$ to $|\psi(0)\rangle$ results in the state $|\psi(NT)\rangle$. Obviously, the eigenstates of $\mathcal{F}$ play an important role,

$$\hat{\mathcal{F}}|\varphi^{(j)}\rangle=e^{-i\varepsilon_j}|\varphi^{(j)}\rangle,\quad j=0,1,\ldots\ . \tag{22}$$

The eigenvalues $\exp(-i\varepsilon_j)$ are complex numbers of modulus one, and the real *quasienergies* ε_j are defined modulo 2π. In general, the nature of the spectrum of quasienergies, be it finite, countable, or continuous, reflects the complexity of the system's dynamics [23]. If the Hamiltonian does not explicitly depend on time, the eigenstates $|\varphi^{(j)}\rangle$ of the Floquet operator $\hat{\mathcal{F}}$ coincide with those of the Hamilton operator $\hat{H}$, that is, $|\varphi^{(j)}\rangle=|\psi^{(j)}\rangle$, while the energy eigenvalues are related to the quasienergies by $\varepsilon_j=(E_jT/\hbar)\mathrm{mod}\ 2\pi$. This association will continue to hold for a weak time-dependent driving force, and the states $|\varphi^{(j)}\rangle$ can be ordered by sorting them according to the size of the expectation value of the energy averaged over one period T:

$$\overline{\langle\hat{H}\rangle}_j=\frac{1}{T}\int_0^T dt\langle\varphi^{(j)}|\hat{U}(t,0)^\dagger\hat{H}(t)\hat{U}(t,0)|\varphi^{(j)}\rangle. \tag{23}$$

As a matter of fact, it is the invariance of the Hamiltonian $\hat{H}(t)$ under the time translation Ξ which leads to the existence of quasienergies defined according to Eq. (22).

B. Generalized parity

For potentials symmetric under spatial reflection, that is, $V_{\mathrm{DW}}(-x)=V_{\mathrm{DW}}(x)$, the Hamiltonian $\hat{H}(t)$ in Eq. (3) is invariant under a simultaneous transformation of space and time,

$$\Pi:x\rightarrow-x\quad\text{and}\quad t\rightarrow t+T/2, \tag{24}$$

known as *generalized parity* [24]. Here, the property of the driving force $g(t+T/2)=-g(t)$ is crucial.

The invariance of the Hamiltonian under the transformation in Eq. (24) has an important consequence for the explicit form of the Floquet operator $\hat{\mathcal{F}}$. The symmetry Π means that

$$\hat{H}(t+T/2)=\hat{P}\hat{H}(t)\hat{P}, \tag{25}$$

where reflection at the origin is described by the parity operator $\hat{P}$,

$$\hat{P}\psi(x)=\psi(-x), \quad \hat{P}^2=1, \quad \hat{P}=\hat{P}^\dagger, \tag{26}$$

and $\mathsf{P}\psi_n=\psi_{-n}$ for the discrete model. Equation (25) implies that the propagator over the *second* half of a period T can be expressed by the propagator over its *first* half:

$$\hat{U}(T,T/2)=\hat{P}\hat{U}(T/2,0)\hat{P}. \tag{27}$$

Hence the propagation over a full period of time can be written as a *square*,

$$\begin{aligned}\hat{U}(T,0)&=\hat{U}(T,T/2)\hat{U}(T/2,0)\\&=\hat{P}\hat{U}(T/2,0)\hat{P}\hat{U}(T/2,0)\end{aligned} \tag{28}$$

or, in terms of the Floquet operator

$$\hat{\mathcal{F}}=\hat{\mathcal{S}}^2, \quad \hat{\mathcal{S}}=\hat{P}\hat{U}(T/2,0). \tag{29}$$

Thus the action of the Floquet operator is given as a twofold application of its ''root'' $\hat{\mathcal{S}}$, being a simple product of propagation over half the period T and a reflection at the origin. This decomposition has not been observed before.

In the following, the eigenequation of $\hat{\mathcal{S}}$,

$$\hat{\mathcal{S}}|\varphi^{(j)}\rangle=e^{-i\sigma_j}|\varphi^{(j)}\rangle, \tag{30}$$

will be studied instead of Eq. (22). The eigenstates of the operators $\hat{\mathcal{S}}$ and $\hat{\mathcal{F}}$ coincide, and the relation between their eigenvalues is governed by

$$\varepsilon_j=2\sigma_j \mod 2\pi. \tag{31}$$

This relation is important for the discussion of degenerate quasienergies ε_j.

C. Generalized time reversal

The third symmetry transformation again involves both space and time:

$$\Theta: x\to -x \quad \text{and} \quad t\to -t. \tag{32}$$

It will be called *generalized time reversal*. The symmetry is a consequence of the property $g(-t)=-g(t)$ of the driving force.

In Hilbert space, the transformation Θ is represented by an antilinear operator $\hat{A}$, not a linear one. There is *no* conserved quantity associated with it. However, it is possible to construct a symmetry-adapted basis such that the (Floquet) eigenfunctions do have a particular structure. This property is a generalization of the possibility to choose purely real eigenfunctions if a system is invariant under time reversal, leading to a real symmetric Hamiltonian. Let us introduce the antilinear operator $\hat{K}$ which has the properties

$$\hat{K}^2=1, \quad \hat{K}=\hat{K}^\dagger, \quad \hat{K}\hat{O}\hat{K}=\hat{O}^*, \tag{33}$$

where $\hat{O}$ is any operator (expressed in the position representation) or a complex number, and the star * denotes complex conjugation. Schrödinger's equation (2) is invariant under the application of the antiunitary operator $\hat{P}\hat{K}$ combined with the reflection of the time parameter

$$\hat{P}\hat{K}\otimes(t\to -t). \tag{34}$$

This follows from the properties of $\hat{K}$ in Eq. (33), the symmetry Θ, and the fact that $\hat{H}(t)$ is real. Hence, if the state $|\psi(t)\rangle$ is a solution of the time-dependent Schrödinger equation, then the transformed state $\hat{P}\hat{K}|\psi(-t)\rangle$ is a solution, too. Similarly, if $|\varphi^{(j)}\rangle$ is an eigenstate of the Floquet operator $\hat{\mathcal{F}}$ (or the operator $\hat{\mathcal{S}}$) according to Eq. (22), then the transformed state $\hat{P}\hat{K}|\varphi^{(j)}\rangle$ is also an eigenstate. This is seen from combining the time periodicity Ξ of the Hamiltonian with the symmetry Θ implying that

$$\hat{H}(T-t)=\hat{P}\hat{H}(t)\hat{P}. \tag{35}$$

Therefore applying parity to the Floquet operator $\hat{\mathcal{F}}=\hat{U}(T,0)$ corresponds to a reversed time ordering. Since the Hamiltonian is real symmetric, $\hat{H}=(\hat{H}^\dagger)^*=\hat{H}^T$, this is identical to a transposition:

$$\begin{aligned}\hat{P}\hat{\mathcal{F}}\hat{P}&=\lim_{N\to\infty} e^{-(i/\hbar)\hat{P}\hat{H}(t_N)\hat{P}\Delta t}\cdots e^{(-i/\hbar)\hat{P}\hat{H}(t_1)\hat{P}\Delta t}\\&=\lim_{N\to\infty} e^{-(i/\hbar)\hat{H}(t_1)\Delta t}\cdots e^{-(i/\hbar)\hat{H}(t_N)\Delta t}=\hat{\mathcal{F}}^T,\end{aligned}$$

using Eqs. (19) and (35); the superscript T denotes the transpose. This property is also shared by the operator $\hat{\mathcal{S}}$,

$$\hat{P}\hat{\mathcal{S}}\hat{P}=\hat{\mathcal{S}}^{\mathrm{T}}. \tag{36}$$

As a consequence one has

$$\begin{aligned}\hat{\mathcal{S}}(\hat{P}\hat{K}|\varphi^{(j)}\rangle)&=\hat{P}\hat{\mathcal{S}}^T\hat{K}|\varphi^{(j)}\rangle=\hat{P}\hat{K}\hat{\mathcal{S}}^\dagger|\varphi^{(j)}\rangle=\hat{P}\hat{K}e^{+i\sigma_j}|\varphi^{(j)}\rangle\\&=e^{-i\sigma_j}(\hat{P}\hat{K}|\varphi^{(j)}\rangle),\end{aligned} \tag{37}$$

where the antilinearity of $\hat{K}$, the unitarity of $\hat{\mathcal{S}}$, and Eq. (36) have been used. Thus the states $|\varphi^{(j)}\rangle$ and $\hat{P}\hat{K}|\varphi^{(j)}\rangle$ are both eigenstates of $\hat{\mathcal{S}}$ with the *same* eigenvalue. They represent the *same* physical state if the eigenvalues $\{e^{-i\sigma_j}\}$ are not degenerate.

Now the definition of a symmetry-adapted basis emerges naturally for systems being invariant under generalized time reversal. Starting with an arbitrary orthonormal basis $\{|\varphi_n\rangle\}$, the symmetry-adapted basis $\{|\Phi_n\rangle\}$ is obtained from a linear combination of the original and transformed states:

$$|\Phi_n\rangle = \begin{cases} c(|\varphi_n\rangle + \hat{P}\hat{K}|\varphi_n\rangle) & \text{if } \hat{P}\hat{K}|\varphi_n\rangle \neq -|\varphi_n\rangle \\ c(|\varphi_n\rangle + i\hat{P}\hat{K}|\varphi_n\rangle) & \text{if } \hat{P}\hat{K}|\varphi_n\rangle = -|\varphi_n\rangle, \end{cases} \tag{38}$$

where c is a real normalization constant. The orthonormal basis states $|\Phi_n\rangle$ have the important property that they are *invariant* under the application of the antiunitary operator $\hat{P}\hat{K}$:

$$\hat{P}\hat{K}|\Phi_n\rangle = |\Phi_n\rangle, \tag{39}$$

using $(\hat{P}\hat{K})^2=1$. This construction of the *normal form* (38) is equivalent to Wigner's general treatment of anti-unitary operators [25], as shown in Appendix A. Expressed in the basis (38) both the root $\hat{\mathcal{S}}$ and the Floquet operator $\hat{\mathcal{F}}$ turn out to be symmetric,

$$\mathcal{S}_{kl} = \langle \Phi_k|\hat{\mathcal{S}}\Phi_l\rangle = \langle \hat{P}\hat{K}\Phi_k|\hat{P}\hat{K}\hat{\mathcal{S}}\Phi_l\rangle^* = \langle \hat{P}\hat{K}\Phi_k|\hat{\mathcal{S}}^\dagger \hat{P}\hat{K}\Phi_l\rangle^*$$
$$= \langle \Phi_k|\hat{\mathcal{S}}^\dagger \Phi_l\rangle^* = \mathcal{S}_{lk}, \tag{40}$$

where the antiunitarity of $\hat{P}\hat{K}$ and Eqs. (36) and (39) have been used. Hence, the operator $\hat{\mathcal{S}}$ belongs to the orthogonal ensemble of symmetric unitary matrices [26,23] obeying

$$\hat{\mathcal{S}} = \hat{\mathcal{S}}^T, \tag{41}$$

when expressed in the symmetry-adapted basis. This result holds generally for systems having an antiunitary symmetry $\hat{A}$ provided $\hat{\Lambda} = \hat{A}^2 = 1$. It has to be emphasized that the relations (36) and (41) are equivalent. However, this equivalence is not a trivial one, since the construction of the symmetry-adapted basis explicitly involves the transformation (38) which is neither unitary nor antiunitary.

Relation (37) and definition (38) lead to the normal form of a Floquet eigenstate $|\Phi^{(j)}\rangle$ in the sense that it is also an eigenstate of the operator associated with generalized time reversal:

$$\hat{P}\hat{K}|\Phi^{(j)}\rangle = |\Phi^{(j)}\rangle. \tag{42}$$

Hence, the corresponding wave functions satisfy

$$\Phi^{(j)}(x) = [\Phi^{(j)}(-x)]^*, \quad x \in \mathbf{R}, \tag{43}$$

which, in the discrete model, reads

$$\Phi_n^{(j)} = [\Phi_{-n}^{(j)}]^*, \quad n = 0, \pm 1. \tag{44}$$

The time-independent functions in Eqs. (43) and (44), respectively, are identical to their complex conjugate reflected at the origin and thus they have the form

$$q(x)e^{iu(x)}, \quad q(x) = q(-x) \quad \text{and} \quad u(x) = -u(-x) \tag{45}$$

and

$$\left(\frac{1}{\sqrt{2}}\, e^{i\phi} \sin\alpha, \cos\alpha, \frac{1}{\sqrt{2}}\, e^{-i\phi} \sin\alpha\right), \quad \alpha, \phi \in [0, 2\pi), \tag{46}$$

respectively. These relations have an analog in the well known case of systems being invariant under (standard) time reversal, where wave functions can be chosen real.

IV. DISCRETE DOUBLE WELL

First, the exact solution of the undriven discrete double well is presented, including a time-independent asymmetry of the well. Various properties of the discrete model are shown to agree qualitatively with those of the continuous one. Then, the discretized version of the *driven* double well is investigated, the focus being on crossings of quasienergies.

A. Undriven system

The discrete system with time-independent asymmetric potential is described by the Hamiltonian

$$\mathsf{H} = \mathsf{H}_0 + \frac{\Delta V}{2} \operatorname{diag}(-1, 0, 1), \tag{47}$$

with H_0 from Eq. (10). The energy eigenvalues E_j in the time-independent Schrödinger equation

$$\mathsf{H}|\psi^{(j)}\rangle = E_j|\psi^{(j)}\rangle, \tag{48}$$

are obtained from the roots of the characteristic polynomial of the matrix H:

$$E_j = \frac{1}{3}\,\mathrm{Tr}\,\mathsf{H}_0 + 2\sqrt{p}\,\cos\left(\varphi + \frac{2(j+1)\pi}{3}\right), \tag{49}$$

where $j=0,1,2$, and

$$\varphi = \frac{1}{3}\arccos(qp^{-3/2}),$$
$$p = \frac{1}{3}\left(\frac{V_B^2}{3} + \frac{(\Delta V)^2}{4} + 2\eta^2\right),$$
$$q = \frac{V_B}{3}\left(\frac{V_B^2}{9} - \frac{(\Delta V)^2}{4} + \eta^2\right). \tag{50}$$

The term $(\mathrm{Tr}\,\mathsf{H}_0)/3$ can be removed by shifting the origin of the energy axis by an amount $V_G = -(6\eta + V_B)/3$, and the new Hamiltonian is given by a traceless matrix. Even for the three-level system, the dependence of the eigenvalues on the barrier height V_B and the asymmetry ΔV is far from trivial. The dependence on ΔV is quadratic throughout since the eigenvalues cannot be sensitive to the transformation $\Delta V \rightarrow -\Delta V$.

The splitting of the two ground states,

$$\Delta E = E_1 - E_0 = 2\sqrt{3p}\,\sin\varphi, \tag{51}$$

is approximately given by

$$\Delta E = \sqrt{(\Delta V)^2 + 4(\eta^2/V_B)^2} + O((\eta^2/V_B)^2). \tag{52}$$

This result is to be compared with the dependence of the tunnel splitting in a continuous double-well potential. A semiclassical calculation [27] leads to an asymptotic expres-

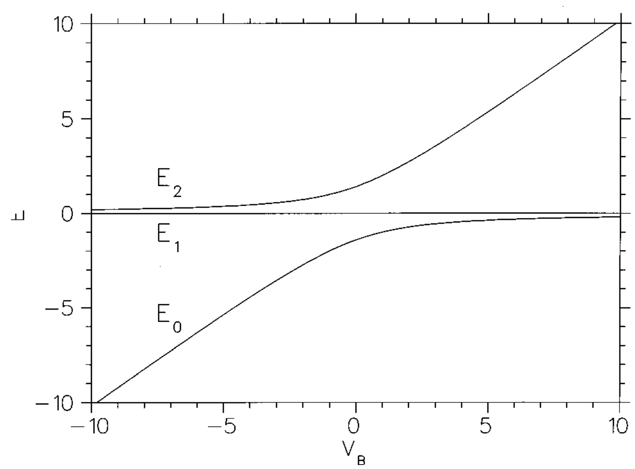

FIG. 2. Eigenvalues of the discrete system as a function of the barrier height V_B for the symmetric double well, $\Delta V=0$. The energy axis has been shifted by an amount $V_G=-2\eta$ so that the barrier-independent eigenvalue is zero, $E_1=0$; units are such that $\eta=1$.

sion similar to Eq. (52): the main difference is that the barrier-dependent term under the root decreases exponentially with V_B, not algebraically. Nevertheless, the overall behavior of the eigenvalues is correct. In Fig. 2, the spectrum is shown for a wide range of values of the barrier height V_B.

For $V_B\rightarrow\infty$, the wells decouple and the energy splitting equals the potential asymmetry $\Delta E=\Delta V$. The eigenstates $\psi^{(0)}$ and $\psi^{(1)}$ are now localized on the left and on the right of the barrier, respectively (provided that $\Delta V\neq 0$), as can be seen from the ‘‘spatial’’ structure of the three-component eigenfunctions of H,

$$\psi_0^{(j)}=\eta^{-1}\left[\eta^{-2}+\left(\frac{V_B}{3}-\frac{\Delta V}{2}+E_j\right)^{-2}+\left(\frac{V_B}{3}-+\frac{\Delta V}{2}+E_j\right)^{-2}\right]^{1/2},$$

$$\psi_{\pm 1}^{(j)}=-\eta\left(\frac{1}{3}V_B\mp\frac{\Delta V}{2}+E_j\right)^{-1}\psi_0^{(j)}. \qquad (53)$$

For finite V_B and a slight asymmetry ΔV, the eigenstates of H are almost symmetric and antisymmetric under reflection at the origin, as illustrated in Fig. 3. For vanishing ΔV, parity $\hat{P}$ is a conserved quantity and the states are (anti)symmetric under spatial inversion, also following from Eq. (53) by taking the limit $\Delta V\rightarrow 0$.

The physical meaning of the splitting ΔE easily emerges from looking at the time evolution of states $\psi^{L,R}(t)$ initially localized in a well: $|\psi_k^L(0)|^2=\delta_{-1,k}$ [or $|\psi_k^R(0)|^2=\delta_{+1,k}$]. For small ΔV the initial states $\psi^{L,R}(0)$ are well approximated by a superposition of the two first eigenstates

$$\psi^{L,R}(0)\approx\frac{1}{\sqrt{2}}(\psi^{(0)}\pm\psi^{(1)}). \qquad (54)$$

The time evolution of this state reads

$$\psi^L(t)\approx\frac{1}{\sqrt{2}}(\psi^{(0)}e^{-iE_0t/\hbar}+\psi^{(1)}e^{-iE_1t/\hbar})$$

$$=\frac{1}{\sqrt{2}}(\psi^{(0)}+\psi^{(1)}e^{-i\Delta Et/\hbar})e^{-iE_0t/\hbar}. \qquad (55)$$

Upon comparison with Eq. (54) the localized states are seen to evolve into each other: $\psi^L\rightarrow\psi^R\rightarrow\psi^L$, with a characteristic tunneling frequency $\Delta E/\hbar$, apart from a physically irrelevant phase.

While the states at the bottom, $\psi^{(0)}$ and $\psi^{(1)}$, agree well with those of the continuous system, the state $\psi^{(2)}$ is centered about the saddle at the origin, $k=0$. This indicates that the present approach is closely related to the method of *tight binding* [28]: the states of an atomic lattice are approximated by superposing wave functions localized at individual atoms. In this sense, the discrete potential should be thought of as providing three local minima at $k=\pm 1$ and $k=0$ instead of two wells and a saddle in between. Apart from this discrepancy, the discrete approximation of a double well indeed reproduces qualitatively the important features of the states at the bottom of the continuous system.

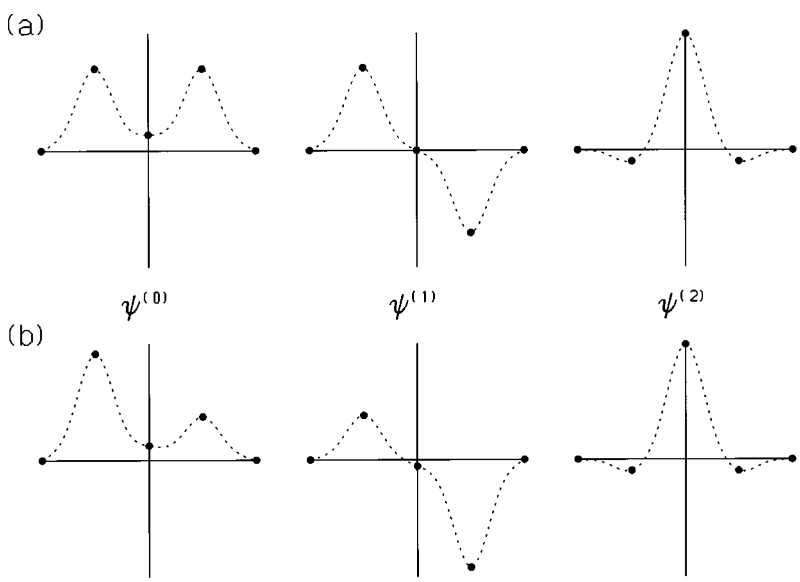

FIG. 3. Eigenstates of the discrete double-well model: (a) symmetric $\Delta V=0$, and (b) asymmetric case $\Delta V=0.2$. The dashed lines are for easy comparison with the wave functions of the continuous double well. Note the *large* value of $\psi^{(2)}$ at the central site (see text).

B. Driven system

This section deals with the eigenvalue equation of the operator S,

$$\mathsf{S}|\varphi^{(j)}\rangle = e^{-i\sigma_j}|\varphi^{(j)}\rangle, \quad j=0,1,2 \tag{56}$$

replacing the Floquet equation (22) since $\mathsf{F}=\mathsf{S}^2$. Explicitly,

$$\mathsf{S}=\mathsf{P}\ \exp(-i\mathsf{H}T/2\hbar), \tag{57}$$

where H is the Hamiltonian with a fixed time-independent asymmetry according to Eq. (47), since the piecewise constant approximation (13) of the driving force $\Delta V(t)$ has been used. Since the operators P and H do *not* commute for an asymmetry $\Delta V \neq 0$, there is no common basis to diagonalize them simultaneously. Only then is the product (57) easily transformed into a single exponential. The operator S can be understood as a product of two finite transformations in the group U(3). Unfortunately, a general Baker-Campbell-Hausdorff formula seems not to be available for this group, although results to *disentangle* exponents have been obtained [29]. This situation is in contrast to the group SU(2) where a closed form for the product is known [30]. Nevertheless, the eigenvalues and the eigenstates of S can be determined analytically, as indicated in Appendix B, since the characteristic polynomial of S is of third order for the discrete model. To this end it is convenient to introduce the operator $\tilde{\mathsf{S}}$ which differs from S by a factor,

$$\tilde{\mathsf{S}}=e^{i\sigma}\mathsf{S}, \quad e^{i\sigma}=(\det \mathsf{S})^{-1/3}=e^{i(\sigma_0+\sigma_1+\sigma_2)/3}. \tag{58}$$

It has unit determinant

$$\det \tilde{\mathsf{S}}=e^{3i\sigma}\det \mathsf{S}=1, \tag{59}$$

so that $\tilde{\mathsf{S}}$ is an element of the group SU(3). The new phases $\tilde{\sigma}_j$ are shifted with respect to the old ones by the amount σ,

$$\tilde{\sigma}_j=(\sigma_j+\sigma) \mod 2\pi, \tag{60}$$

having the property

$$\tilde{\sigma}_0+\tilde{\sigma}_1+\tilde{\sigma}_2=0 \mod 2\pi. \tag{61}$$

Explicitly, in terms of $\tilde{\mathsf{S}}$, one obtains the characteristic polynomial:

$$\lambda^3-(\mathrm{Tr}\ \tilde{\mathsf{S}})\lambda^2+\chi\lambda-\det\tilde{\mathsf{S}}=0. \tag{62}$$

The coefficient χ is easily determined: It follows from the unitarity of $\tilde{\mathsf{S}}$ that its eigenvalues are complex numbers of modulus one, hence the inverse of λ equals its complex conjugate, $\lambda^{-1}=\lambda^*$. When taking into account Eq. (59) one finds that the characteristic polynomial is invariant under a multiplication with $-\lambda^{-3}$ and subsequent complex conjugation. A comparison of coefficients then reveals that $\chi=(\mathrm{Tr}\ \tilde{\mathsf{S}})^*$. Thus the characteristic polynomial reads

$$\lambda^3-(\mathrm{Tr}\ \tilde{\mathsf{S}})\lambda^2+(\mathrm{Tr}\ \tilde{\mathsf{S}})^*\lambda-1=0. \tag{63}$$

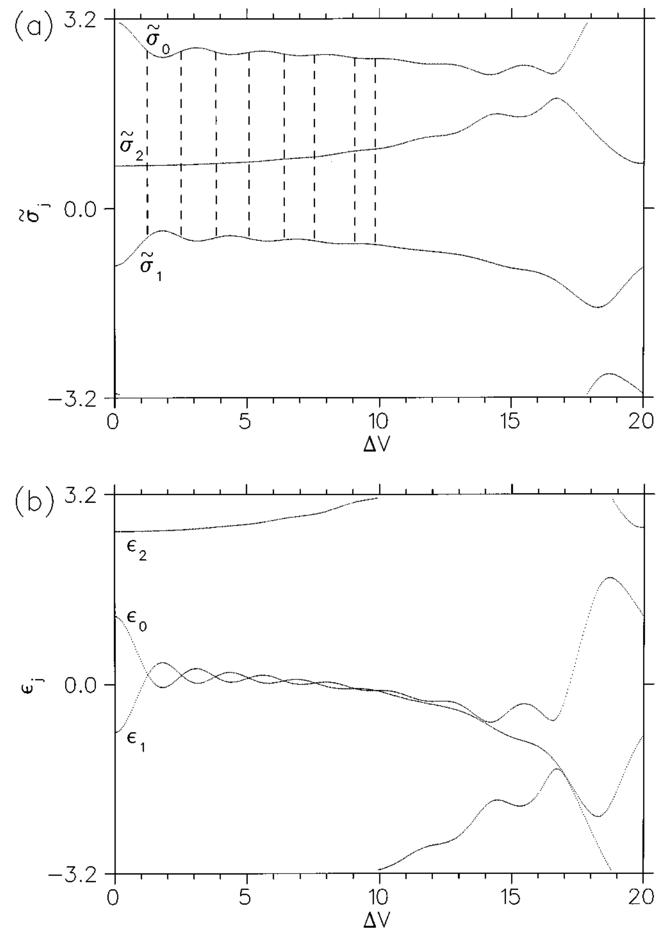

FIG. 4. Eigenphases of the driven discrete driven double-well model for $T=10$, $V_B=10$ for variable asymmetry ΔV: (a) eigenphases $\tilde{\sigma}_j$, (b) quasienergies ε_j, $j=0,1,2$. The dashed vertical lines all have a length of π (see text).

A simple expression for the trace of $\tilde{\mathsf{S}}$ follows if it is evaluated in the eigenbasis of the Hamiltonian H in Eq. (47) of the undriven system:

$$\begin{aligned}\mathrm{Tr}\ \tilde{\mathsf{S}}&=\sum_{j=1}^{3}\langle\psi^{(j)}|e^{i\sigma}\mathsf{P}\ \exp(-i\mathsf{H}T/2\hbar)|\psi^{(j)}\rangle\\&=\sum_{j=1}^{3}e^{-i(E_jT/2\hbar-\sigma)}\langle\psi^{(j)}|\mathsf{P}|\psi^{(j)}\rangle,\end{aligned} \tag{64}$$

containing the expectation values of the parity operator. Its modulus obviously has the property $|\mathrm{Tr}\ \tilde{\mathsf{S}}|\leq 3$. Explicit expressions of the eigenphases $\tilde{\sigma}_j$ of $\tilde{\mathsf{S}}$ are not illuminating due to their involved dependence on the parameters of $\tilde{\mathsf{S}}$ via its trace in Eq. (64). In Fig. 4, the eigenphases $\tilde{\sigma}_j$ and the quasienergies ε_j, respectively, are plotted as functions of the strength of the asymmetry while keeping the period T and the barrier height fixed. The quantities $\tilde{\sigma}_j$ vary with a degree of complexity as a function of ΔV which is surprising in light of the simplicity of the underlying model. They do *not* cross each other which, however, does not exclude the degeneracy of quasienergies. Before turning to the discussion of degenerate quasienergies, a geometric interpretation of the condition

$$\mathrm{Tr}\ \tilde{\mathsf{S}}=e^{-i\tilde{\sigma}_1}+e^{-i\tilde{\sigma}_2}+e^{-i\tilde{\sigma}_3}=z \tag{65}$$

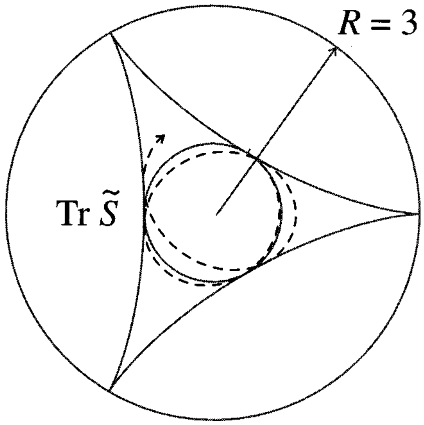

FIG. 5. The behavior of Tr $\tilde{\mathsf{S}}$ in the complex plane for fixed asymmetry ($\Delta V=0.2$) and varying period T is depicted by the dashed line. The quasienergies ε_j do cross if (a) $\tilde{\sigma}_j=\tilde{\sigma}_k$ mod 2π, Tr $\tilde{\mathsf{S}}$ falls on the threefold cycloid $z=\exp(2i\phi)+2\exp(-i\phi)$, $\phi \in[0,2\pi)$, or if (b) $\tilde{\sigma}_j=\tilde{\sigma}_k+\pi$ mod 2π, Tr $\tilde{\mathsf{S}}$ falls on the unit circle, $|z|=1$. The dashed line does *not* touch the cycloid since crossings of the phases $\tilde{\sigma}_j$ are generically avoided.

will be given. Consider $z=Re^{i\alpha}$ as a vector in the complex plane with length $|z|<3$. For a solution of Eq. (65) one has to find three unit vectors $\{e^{-i\tilde{\sigma}_j}\}$ which must combine to give the vector z. This is always possible for a one-parameter family of angles $\{\tilde{\sigma}_j\}$. Then one has to select that particular solution which leads to the correct value of det $\tilde{\mathsf{S}}=1$.

C. Coherent destruction of tunneling

In this section we focus our interest on crossings of quasienergies. They are related to the effect of coherent destruction of tunneling [15], corresponding to the relocalization of a tunneling state in a potential well at stroboscopic times $t=T,2T,\ldots$. As seen in Fig. 4, the quasienergies ε_j may degenerate for specific values of the asymmetry *without* a crossing of the phases $\tilde{\sigma}_j$ (or σ_j). More precisely, the relation between σ_j and ε_j implies that quasienergies do cross if one of the following conditions is fulfilled: (a) $\tilde{\sigma}_j=\tilde{\sigma}_k$ mod 2π, or (b) $\tilde{\sigma}_j=(\tilde{\sigma}_k+\pi)$ mod 2π. These conditions also apply to the original phases σ_j. Generically, in order to realize condition (a), which means to have a degenerate eigenvalue $\exp(-i\tilde{\sigma}_j)$, the variation of *two* system parameters is required. This follows from the fact that the matrix S belongs to the orthogonal ensemble of symmetric unitary matrices according to Eq. (41) and a comparison of the number of free parameters for orthogonal matrices in the nondegenerate and degenerate case [31], respectively. A geometric interpretation of the realization of condition (a) is obtained from looking at the corresponding expression for the trace of S,

$$\mathrm{Tr}\ \tilde{\mathsf{S}}=e^{2i\tilde{\sigma}_j}+2e^{-i\tilde{\sigma}_j}. \tag{66}$$

The right-hand side of Eq. (66) describes a one-dimensional curve in the complex plane, shown in Fig. 5. In the generic case, the complex number $z=\mathrm{Tr}\ \tilde{\mathsf{S}}$ is confined to remain inside the region defined by Eq. (66).

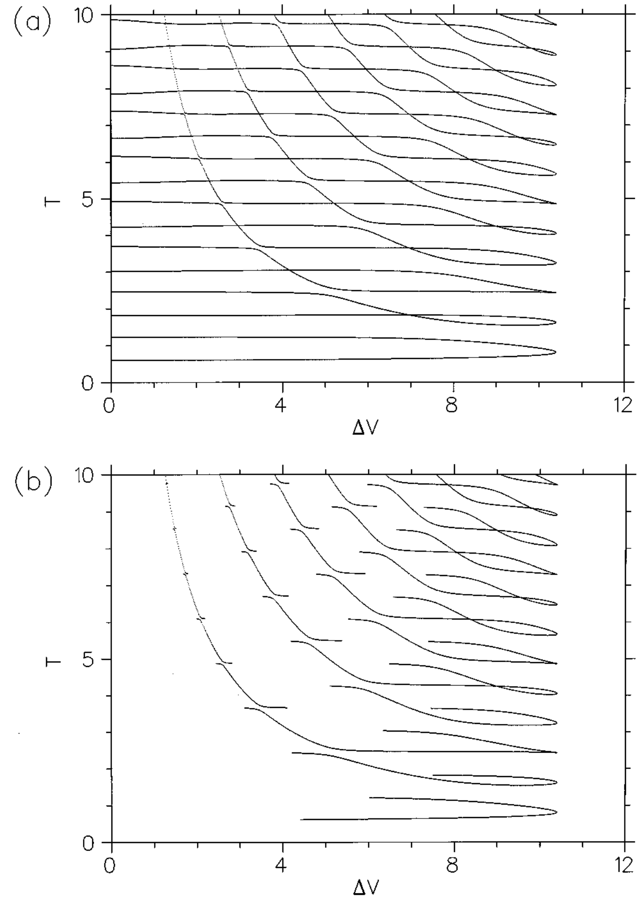

FIG. 6. (a) Contour plot of $|\mathrm{Tr}\ \tilde{\mathsf{S}}|=1$ for the parameters $0\leqslant\Delta V\leqslant 12$ and $0\leqslant T\leqslant 10$, showing the parameter values where a crossing of quasienergies occurs. (b) Same as (a) restricted to the points where the degenerate Floquet eigenstates $|\varphi^{(j)}\rangle,|\varphi^{(k)}\rangle$ are strongly localized in the potential wells.

In order to realize condition (b) the variation of a *single* parameter is sufficient. In this case the difference between two phases $\tilde{\sigma}_0$ and $\tilde{\sigma}_1$, say, has to be an odd multiple of π. In terms of the trace of the root operator this implies that Tr $\tilde{\mathsf{S}}=\exp(-i\tilde{\sigma}_2)$ or

$$|\mathrm{Tr}\ \mathsf{S}|=|\mathrm{Tr}\ \tilde{\mathsf{S}}|=1, \tag{67}$$

describing the unit circle in the complex plane (cf. Fig. 5). In fact, a realization of Eq. (67) is already sufficient in order to fulfill condition (b) as follows from writting down explicitly the roots λ_j of Eq. (63),

$$\{\lambda_j\}=\{\mathrm{Tr}\ \tilde{\mathsf{S}},\pm\sqrt{-(\mathrm{Tr}\ \tilde{\mathsf{S}})^*}\}. \tag{68}$$

Therefore condition (67) determines the locus of degenerate quasienergies. Variation of a single parameter, $0\leqslant T\leqslant 3$, indeed leads to such crossings as shown by the dashed line in Fig. 5, intersecting repeatedly the unit circle.

It is straightforward to obtain a global picture of the presence of crossings in the parameter plane $(\Delta V,T)$, say. Figure 6(a) shows the contours of Tr $\tilde{\mathsf{S}}$ with modulus equal to unity. Obviously, variation of a *single* parameter in the $(\Delta V,T)$ plane will generically result in a crossing, as has been argued before. The arrangement of lines in Fig. 6(a) for not too large values of ΔV is a consequence of condition (b). In the symmetric case, $\Delta V=0$, the operator P is diagonal in the eigenbasis $|\varphi^{(j)}\rangle$ of the nondriven system. For small ΔV, the Floquet eigenstates $|\varphi^{(j)}\rangle$ are well approximated by the states $|\psi^{(j)}\rangle$. Using Eqs. (48), (56), and the definition of P from Eq. (26), one obtains approximate values of the eigenphases,

$$\widetilde{\sigma}_j \approx \left\{ \frac{1}{2} E_j T/\hbar + \sigma + \begin{pmatrix} 0, & j \ \text{even} \\ \pi, & j \ \text{odd} \end{pmatrix} \right\} \mod 2\pi. \quad (69)$$

This equation together with condition (b) approximately reproduces the equidistant lines visible in Fig. 6(a). However, a crossing of these contour lines is avoided, because it would imply a realization of condition (a). The lines almost parallel to the ΔV axis correspond to the (alternating) realizations of condition (b), $\widetilde{\sigma}_j=(\widetilde{\sigma}_k+\pi) \mod 2\pi$ for $(j,k)=(0,2)$ and $(j,k)=(1,2)$. The ''hyperbolic'' branches correspond to the realizations of the same condition for $(j,k)=(0,1)$. From $\widetilde{\sigma}_1-\widetilde{\sigma}_0 \approx \Delta E T/2\hbar+\pi$ one finds the estimate

$$T \approx \frac{4\pi n \hbar}{\Delta E}, \quad n \in \mathbb{Z}. \quad (70)$$

This reproduces the quasihyperbolic behavior $T \propto \Delta V^{-1}$ visible in Fig. 6, since $\Delta E \approx \Delta V$ for sufficiently large asymmetry ΔV.

Another interesting feature appearing in Fig. 6 is the complete disappearance of quasi-energy crossings for large driving amplitude $\Delta V \gtrsim V_B$. This effect is a consequence of the few-level discrete approximation scheme and it has no analog in the continuous system. From Eq. (64) one estimates $|\mathrm{Tr}\ \tilde{\mathsf{S}}| \leq \Sigma |\langle \mathsf{P} \rangle_j|$, where $\langle \mathsf{P} \rangle_j$ is the expectation value of the parity operator for the jth eigenstate of the asymmetric double well [cf. Eq. (53)]. For nonvanishing asymmetry ΔV the modulus of $\langle \mathsf{P} \rangle_j$ is less than unity since parity is not a conserved quantity in this case. At some threshold value $|\mathrm{Tr}\ \tilde{\mathsf{S}}|$ becomes strictly smaller than one, hence the condition (67) necessary for degenerate quasienergies cannot hold for large driving amplitude.

Being mainly interested in the tunneling behavior of states being localized in the wells, we now focus our attention on the quasienergy crossings of these states. Figure 6(b) shows an appropriately modified version of Fig. 6(a), containing only crossings with sufficiently large localization of the corresponding Floquet eigenstates in the wells: the localization probability of the (degenerate) eigenstates $|\varphi^{(j)}\rangle,|\varphi^{(k)}\rangle$ is required to be greater than that of the third eigenstate $|\varphi^{(l)}\rangle$,

$$|\varphi^{(j)}_{\pm 1}|^2, |\varphi^{(k)}_{\pm 1}|^2 \geq |\varphi^{(l)}_{\pm 1}|^2. \quad (71)$$

For $4 \lesssim \Delta V \lesssim 10.5$, not only the ''hyperbolic'' branches in Fig. 6(b) (associated with the resonances $T=4\pi n/\Delta E$ in the undriven system) induce crossings of quasienergies, but the other branches contribute as well. It appears that a crossing of quasienergies enhances the localization probability of the corresponding Floquet eigenstates in the wells.

D. The effective Hamiltonian

In this section we give an alternative physical interpretation of the effect of the time-dependent driving term in the discrete model. To this end we introduce an effective, time-independent Hamiltonian $\mathsf{H}^{\mathrm{eff}}$ producing the same stroboscopic dynamics of the driven system as does the Hamiltonian $\mathsf{H}(t)$ in Eq. (9), i.e.,

$$\mathsf{F}=\exp\left(-\frac{i}{\hbar}\mathsf{H}^{\mathrm{eff}}T\right). \quad (72)$$

Knowledge of a Baker-Campbell-Hausdorff relation for SU(3) would explicitly provide the operator $\mathsf{H}^{\mathrm{eff}}$ in terms of H_0 and P. Since $\mathsf{H}^{\mathrm{eff}}$ is the logarithm of F, it is not uniquely defined in general. However, a natural choice of $\mathsf{H}^{\mathrm{eff}}$ is to have it coincide with the symmetric Hamiltonian H_0 in Eq. (10) in the nondriven case for vanishing asymmetry ΔV,

$$\mathsf{H}^{\mathrm{eff}}=\mathsf{H}_0+\Delta\mathsf{H}, \quad (73)$$

where $\Delta\mathsf{H}=0$ if $\Delta V=0$. Because of its invariance under generalized time reversal, the effective Hamiltonian commutes with the antiunitary operator PK,

$$[\mathsf{H}^{\mathrm{eff}},\mathsf{PK}]=0. \quad (74)$$

In this picture, the eigenvalues E_j^{eff} are related to the quasienergies by $\varepsilon_j=(E_j^{\mathrm{eff}}T/\hbar) \mod 2\pi$, and a crossing corresponds to a resonance

$$(E_j^{\mathrm{eff}}-E_k^{\mathrm{eff}})T/\hbar=2\pi n, \quad j\neq k, n\in\mathbf{Z}. \quad (75)$$

Express the effective Hamiltonian in the basis of the fundamental representation [32] of SU(3),

$$\mathsf{H}^{\mathrm{eff}}=\alpha_0 1+\sum_{k=1}^{8}\alpha_k\lambda_k \quad (\alpha_k \ \text{real}), \quad (76)$$

with the traceless Gell-Mann (3×3) matrices λ_k, the generators of the group. Due to the invariance under generalized time reversal (74) the nine real coefficients α_k are not independent:

$$\alpha_1=\alpha_6, \quad \alpha_2=\alpha_7, \quad \text{and} \ \alpha_3=-\sqrt{3}\alpha_8. \quad (77)$$

Thus six parameters α_k completely characterize both the effective Hamiltonian $\mathsf{H}^{\mathrm{eff}}$ and the Floquet matrix F. In the nondriven case, $\mathsf{H}^{\mathrm{eff}}=\mathsf{H}_0$, the coefficients α_k are related to the system parameters by $\alpha_0=2\eta+V_G+V_B/3$, $\alpha_1=-\eta$, $\alpha_3=-\eta/2+V_B$, $\alpha_2=\alpha_4=\alpha_5=0$.

Let us determine the explicit form of the effective Hamiltonian in the high-frequency limit $T\to 0$. From Eq. (27) and the driving approximation (13) we have that

$$\exp\left(-\frac{i}{\hbar}\mathsf{H}^{\mathrm{eff}}T\right)=\exp\left(-\frac{i}{2\hbar}\mathsf{PHP}T\right)\exp\left(-\frac{i}{2\hbar}\mathsf{H}T\right), \quad (78)$$

with H from Eq. (47). Using the Baker-Campbell-Hausdorff formula [33]

$$e^A e^B = e^{A+B+(1/2)[A,B]-(1/12)([A,[A,B]]+[[A,B],B])+\cdots} \quad (79)$$

one finds the expansion

$$\Delta \mathsf{H} = \eta \frac{\Delta VT}{8\hbar} (\lambda_2 + \lambda_7) + \eta \frac{(\Delta V\, T)^2}{48\hbar^2} (\lambda_1 + \lambda_6) + O((\Delta VT/\hbar)^3), \tag{80}$$

which respects the antiunitary symmetry (77). An explicit calculation of the energy eigenvalues of the effective Hamiltonian (along the same lines as in Sec. IV A) results in an expression for the tunnel splitting ΔE similar to Eq. (51): one only has to replace

$$\eta^2 \rightarrow (\eta^{\text{eff}})^2 = \eta^2[1 + (\Delta VT)^2/40\hbar^2] + O((\Delta VT/\hbar)^3). \tag{81}$$

Consequently, in the limit of high frequency ($T \rightarrow 0$) the driving force effectively *decreases* the height of the potential barrier,

$$V_B^{\text{eff}} \approx V_B[1 - (\Delta VT)^2/40\hbar^2] < V_B, \quad T \rightarrow 0. \tag{82}$$

A decrease of the effective potential barrier in the high-frequency limit is also found for the continuous system [7].

V. SUMMARY

In this paper, we have studied a simple discrete model of a periodically driven particle in a double-well potential. Taking into account all the relevant symmetries, time periodicity, generalized parity, and generalized time reversal, a natural decomposition of the Floquet operator and an associated normal form of its eigenfunctions has been presented. A discussion of quasienergy crossings, motivated by the effect of coherent destruction of tunneling, has revealed that they are closely related to the resonances of the non-driven asymmetric double-well system. The results qualitatively agree to a large extent with those of the continuous model, as far as they are known.

ACKNOWLEDGMENTS

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APPENDIX A: NORMAL FORM OF ANTIUNITARY OPERATORS

Any antiunitary operator $\hat{A}$ can be written as

$$\hat{A} = \hat{U}\hat{K}, \tag{A1}$$

where $\hat{K}$ is complex conjugation (in some basis) and $\hat{U}$ is unitary [25]. Eigenstates of $\hat{A}$ are constructed from those of the unitary operator $\hat{\Lambda} = \hat{A}^2 = \hat{U}\hat{U}^*$. If $|v_0\rangle$ is an eigenstate of $\hat{\Lambda}$ with eigenvalue 1,

$$\hat{\Lambda}|v_0\rangle = |v_0\rangle, \tag{A2}$$

then the state

$$|v\rangle = \begin{cases} c(|v_0\rangle + \hat{A}|v_0\rangle) & \text{if } \hat{A}|v_0\rangle \neq -|v_0\rangle \\ c(|v_0\rangle + i\hat{A}|v_0\rangle) & \text{if } \hat{A}|v_0\rangle = -|v_0\rangle \end{cases} \tag{A3}$$

is obviously an eigenstate of the operator $\hat{A}$. In the context of this paper, the operator $\hat{U}$ in Eq. (A1) is to be identified [cf. Eq. (34)] with spatial reflection $\hat{P}$, and the operator $\hat{\Lambda}$ equals the identity. If the eigenvalue of the eigenstate $|v_0\rangle$ of $\hat{\Lambda}$ is different from one, the operator $\hat{A}$ does not have eigenstates—instead a set of ‘‘characteristic vectors’’ [25] can be associated with it.

APPENDIX B: ANALYTICAL DETERMINATION OF S AND ITS EIGENVALUES

The knowledge of the eigenbasis and eigenvalues of the time-independent asymmetric Hamiltonian H in Eq. (9) allows one to explicitly calculate the matrix S defined in Eq. (57):

$$\mathsf{S} = \mathsf{P}\mathsf{U}\ \text{diag}(e^{-iE_0T/2\hbar}, e^{-iE_1T/2\hbar}, e^{-iE_2T/2\hbar})\mathsf{U}^\dagger. \tag{B1}$$

Here, the unitary transformation U is composed of the eigenbasis of H, Eq. (53): $\mathsf{U}_{jk} = \psi_{k-2}^{(j-1)}$, $j,k = 1,2,3$. The eigenvalues E_j of H are inserted from Eq. (49). In principle, one can obtain the eigenvalues $\exp(-i\sigma_j)$ and eigenstates in closed form, since the characteristic polynomial of S in Eq. (63) is of third order. However, we refrain from presenting the explicit results because the rather involved expressions do not offer much insight.

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